EFFECT OF HIGH-FREQUENCY VIBRATION ON FILTRATION CONVECTION

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Below we will study the effect of rapid vibration on development of convection in a porous medium. The problem of filtration convection was first considered in [1]. A review of subsequent studies can be found in [2-4]. The action of high frequency vibration on convection development in a liquid (Oberbeck–Boussinesq equation) was first studied in [5], where a closed dynamic system was derived for the average thermohydrodynamic field and the stabilizing effect of vertical oscillations was established. An exact mathematical justification of the averaging method for the problem of convection within a field of rapidly oscillating forces was given in [6].

In [7] the effects of high frequency vibration were confirmed experimentally in a study of convection in a homogeneous medium. It is hoped that the present study of convection in a homogeneous medium. It is hoped that the present study will also offer grounds for experiment.

Below we will derive averaged equations of filtration convection for an arbitrary region. Analysis of the stability of relative equilibrium was performed for a planar horizontal layer on the rigid boundaries of which a constant temperature is maintained. An interesting feature of the linearized system is the fact that as a rule it lacks coefficients which vary vertically. A system with constant coefficients is obtained only for vertical oscillations or for the condition b = m ($b = (\rho c_p)_m/(\rho c_p)_\ell$, the ratio of the specific heats; m, porosity). For these cases it has been proved that convection development is produced by monotonic disturbances (the monotonicity principle). The effect of vertical oscillations has been studied in detail. It has been found that as for a homogeneous liquid, sufficiently intense vibration completely suppresses convection (absolute stabilization).

1. Formulation of the Problem. The vessel containing the porous medium saturated by a viscous incompressible liquid performs harmonic oscillations along a specified direction $\mathbf{s} = (\cos\varphi, \sin\varphi)$ following a law $a/\Omega\cos\Omega t$. The temperature distribution is specified on the boundary, which is assumed rigid, impermeable, and ideally thermally conductive. We will make use of the filtration convection equations in the Oberbeck–Boussinesq approximation obtained in [8]. Transforming therein to a moving coordinate system fixed to the vessel we obtain

$$\frac{1}{m}\left(\frac{\partial\mathbf{u}}{\partial t}+\frac{1}{m}(\mathbf{u},\nabla)\,\mathbf{u}\right)=-\frac{\nabla p}{\rho_\ell}+g\beta T\gamma-\frac{\nu}{K}\mathbf{u}+\mathbf{w}_e\beta T,\ \operatorname{div}\mathbf{u}=0,$$
$$(\rho c_p)_{\text{cp}}\frac{\partial T}{\partial t}=\varkappa_{\text{m}}\Delta T-(\rho c_p)_\ell(\mathbf{u},\nabla T), \tag{1.1}$$
$$\mathbf{w}_e=-a\Omega\cos\Omega t\mathbf{s},\ \mathbf{s}=(\cos\varphi,\ \sin\varphi),$$

where $\mathbf{u}$ is the relative filtration velocity, T is temperature, p is the convective pressure, ρ is the density, g is the acceleration of gravity, γ is a unit vector directed vertically upward, m is the porosity coefficient, K is the permeability, ν, β, κ are kinematic viscosity, volume expansion, and thermal conductivity coefficients, c_p is the specific heat (subscripts ℓ and m denote the liquid and porous medium, respectively), a is the vibration rate, Ω is the frequency, φ is the angle at which the vibration is directed relative to the horizontal plane, such that $\varphi = 0$ corresponds to horizontal oscillations and $\varphi = \pi/2$, to vertical.

We note that the transition in the filtration convection equations to the moving coordinate system is performed just as in the case of convection in a homogeneous medium [5], the only difference being replacement of the Newtonian viscosity by Darcy viscosity.

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In system (1.1) we transform to dimensionless variables choosing the following units of measurement: length ℓ, time ℓ^2/ν, velocity ν/ℓ, temperature $A\ell$, pressure $\rho_\ell \nu^2/K$, obtaining

$$c\frac{\partial \mathbf{u}}{\partial t}+\frac{c}{m}(\mathbf{u},\nabla)\mathbf{u}=-\nabla p+\boldsymbol{\gamma}\,\mathrm{Gr}\,T-\mathbf{u}-G(\cos\omega t)\,\mathbf{s}T,\ \operatorname{div}\mathbf{u}=0,$$
$$b\frac{\partial T}{\partial t}=\frac{1}{\mathrm{Pr}}\Delta T-(\mathbf{u},\nabla T) \tag{1.2}$$

(the dimensionless unknowns are denoted by the same letters as the dimensional ones). System (1.2) contains the dimensionless parameters: $c = K/(\ell^2 m)$, the ratio of dimensionless permeability and porosity; $\mathrm{Pr} = \nu/\chi$, the Prandtl number [$\chi = \kappa_m/(\rho c_p)_\ell$ is the thermal diffusivity of the medium]; $\mathrm{Gr} = (gA\beta\ell^2 K)/\nu^2$, the Grashof number; $G = (a\Omega\beta A\ell^2 K)/\nu^2$, the vibration parameter; $\omega = \Omega\ell^2/\nu$, the dimensionless frequency; $b = (\rho c_p)_m/(\rho c_p)_\ell$, the specific heat ratio. On the boundary S the normal component of heat filtration rate vanishes and the temperature is specified:

$$u_n = 0,\ T = T_0. \tag{1.3}$$

The initial conditions for system (1.2) consist of specification at $t = 0$ of the velocity $\mathbf{u}$ and temperature T. We will now consider the high frequency asymptote ($\omega \to \infty$) of an arbitrary solution for the usual assumptions of the averaging method.

2. Derivation of Averaged Equations. We will consider vibration of high frequency and low amplitude ($\omega \to \infty$), assuming that the modulation rate remains finite. We apply the averaging method in Kapitsa's form to system (1.2), (1.3) as was done in [5, 9] for a homogeneous liquid. We will seek the unknowns $\mathbf{u}$, T, p in the form

$$\mathbf{u}=\mathbf{v}+\boldsymbol{\xi},\ T=\tau+\eta,\ p=q+\delta, \tag{2.1}$$

where $\mathbf{v}$, τ, q are slow components and ξ, η, δ are rapid ones, having a zero average over time:

$$\boldsymbol{\xi}=-\frac{G}{\omega c}(\sin\omega t)\,\mathbf{w},\ \mathbf{w}=\Pi(\mathbf{s}\tau),$$
$$\operatorname{div}\mathbf{w}=0,\ \eta=-\frac{G}{\omega^2 cb}(\cos\omega t)(\mathbf{w},\nabla\tau),\ w_n|_S=0. \tag{2.2}$$

Here the operator Π is an orthoprojector in L_2 in the subspace of solenoidal vectors with a normal component equal to zero on the boundary. In other words, $\mathbf{w}=\mathbf{s}\tau-\nabla\Phi$, the function Φ is a solution of the Neiman problem:

$$\Delta\Phi=(\nabla\tau,\ \mathbf{s}),\ \partial\Phi/\partial n|_S=(\mathbf{s}\tau,\ \mathbf{n}) \tag{2.3}$$

($\mathbf{n}$ is a unit vector in the direction of the external normal). Equations (2.2), (2.3) express the rapid components ξ, η in terms of the slow temperature component τ. They can be obtained in a natural manner by substituting Eq. (2.1) in the original Eqs. (1.2) and separating the main vibration terms. Substituting Eq. (2.1) in Eqs. (1.2) and (1.3) and averaging over the explicitly appearing time, we obtain a closed dynamic system for the average thermohydrodynamic field:

$$c\frac{\partial \mathbf{v}}{\partial t}+\frac{c}{m}(\mathbf{v},\nabla)\mathbf{v}=-\nabla q+\boldsymbol{\gamma}\,\mathrm{Gr}\,\tau-\mathbf{v}+\mathrm{Gv}(\mathbf{w},\nabla)\left(\frac{m}{b}\mathbf{s}\tau-\mathbf{w}\right),\ \operatorname{div}\mathbf{v}=0,$$
$$b\frac{\partial\tau}{\partial t}=\frac{1}{\mathrm{Pr}}\Delta\tau-(\mathbf{v},\nabla\tau),\ \operatorname{div}\mathbf{w}=0, \tag{2.4}$$
$$\Delta\mathbf{w}=-\operatorname{rot}\operatorname{rot}(\mathbf{s}\tau),\ v_n=w_n=0,\ \tau=T_0\ \text{on}\ S.$$

Time does not appear explicitly in Eq. (2.4), but additional terms have appeared with the coefficient $\mathrm{Gv} = a^2\beta^2A^2\ell^2K/(2\nu^2)$, the vibration Grashof number for the porous medium.

3. Mechanical Equilibrium. It follows from system (2.4) that an average filtration flow is absent ($\mathbf{v}_0 = 0$) if the specified region, heating conditions and vibration direction are such that the equations

$$\mathrm{Gr}\,\nabla\tau_0\times\boldsymbol{\gamma}+\mathrm{Gv}\operatorname{rot}\left[(\mathbf{w}_0,\nabla)\left(\frac{m}{b}\mathbf{s}\tau_0-\mathbf{w}_0\right)\right]=0,$$
$$\Delta\tau_0=0,\ \operatorname{rot}\mathbf{w}_0=\nabla\tau_0\times\mathbf{s},\ \operatorname{div}\mathbf{w}_0=0,\ w_{0n}=0,\ \tau_0=T_0\ \text{on}\ S \tag{3.1}$$

are satisfied.

The conditions of Eq. (3.1) are necessary for existence of mechanical equilibrium, and are sufficient for a single bond region. For a homogeneous liquid Eq. (3.1) has not been solved in the general case; in [10, 11] some equilibrium configurations for weightlessness (Gr = 0) were considered. We will not study the problem of Eq. (3.1) further, noting only that if $\nabla\tau_0 \parallel \mathbf{s}$, then $\mathbf{w}_0 = 0$ and it follows from Eq. (3.1) that $\nabla\tau_0 \parallel \gamma$ and $\tau_0 = -Cy + D$.

4. Stability of Mechanical Equilibrium. We will now study development of convection in a planar horizontal layer ($|y| \le \ell/2$) on the boundary of which temperatures T_1 and T_2 are specified such that the temperature gradient $A = (T_1 - T_2)/\ell$. The problem of Eq. (2.3) then has the following equilibrium solution:

$$\mathbf{v}_0 = 0,\ \tau_0 = -y + B,\ q_0 = \mathrm{Gr}(-y^2/2 + By) + \text{const},$$
$$w_{0x} = -(\cos\varphi)y,\ w_{0y} = 0,\ B = (T_1 + T_2)/(2(T_1 - T_2)).$$

For small perturbations we write the linearized system in the form

$$\begin{gathered} c\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \gamma\,\mathrm{Gr}\,T - \mathbf{u} + \mathrm{Gv}\left[(\mathbf{w}_0, \nabla)\left(\frac{m}{b}\,\mathbf{s}T - \mathbf{w}\right) + \right. \\ \left. + (\mathbf{w}, \nabla)\left(\frac{m}{b}\,\mathbf{s}\tau_0 - \mathbf{w}_0\right)\right],\ \operatorname{div}\mathbf{u} = 0, \\ b\frac{\partial T}{\partial t} = \frac{1}{\mathrm{Pr}}\Delta T - (\mathbf{u}, \nabla\tau_0)\ \operatorname{div}\mathbf{w} = 0, \\ \Delta\mathbf{w} = -\operatorname{rot}\operatorname{rot}(\mathbf{s}T),\ u_y = w_y = T = 0\ \text{ for } y = \pm 1/2. \end{gathered} \tag{4.1}$$

In the general case system (4.1) contains the variable coefficient $w_{0x}(y)$. After transformations that quantity reduces to the form

$$\begin{gathered} c\frac{\partial\Delta u_y}{\partial t} = \mathrm{Gr}\frac{\partial^2 T}{\partial x^2} - \Delta u_y + \mathrm{Gv}\left[\left(\frac{m}{b} - 1\right) w_{0x}\frac{\partial\Delta w_y}{\partial x} + \right. \\ \left. + \frac{m}{b}\left(\cos^2\varphi\frac{\partial^2 T}{\partial x^2} - \sin\varphi\frac{\partial^2 w_y}{\partial x^2} + \cos\varphi\frac{\partial^2 w_y}{\partial x\,\partial y}\right)\right], \\ b\frac{\partial T}{\partial t} = \frac{1}{\mathrm{Pr}}\Delta T + u_y,\ \Delta w_y = \frac{\partial^2 T}{\partial x^2}\sin\varphi - \frac{\partial^2 T}{\partial x\,\partial y}\cos\varphi, \\ u_y = w_y = T = 0\ \text{ for } y = \pm 1/2. \end{gathered} \tag{4.2}$$

System (4.2) has constant coefficients if $w_{0x} = 0$ or $b = m$.

5. Monotonicity Principle. We will show that this principle is satisfied for $b = m$, $0 \le \varphi \le \pi/2$ or for $\varphi = \pi/2$ and any b, m. In system (4.2) we eliminate the function u_y and consider normal perturbations

$$T(x, y, t) = \exp(\sigma t + i\alpha x)\theta(y),$$
$$w_y(x, y, t) = \exp(\sigma t + i\alpha x)w(y).$$

For the amplitude $\theta(y)$, $w(y)$ we obtain

$$\begin{gathered} \mathrm{Pr}cb\sigma^2 L\theta + \sigma(\mathrm{Pr}bL\theta - cL^2\theta) - L^2\theta + \alpha^2(\mathrm{R} + \mu\cos^2\varphi)\theta = \\ = \mu(\alpha^2\sin\varphi w + i\alpha\cos\varphi Dw), \\ -Lw = \alpha^2\sin\varphi\theta + i\alpha\cos\varphi D\theta, \\ w = \theta = L\theta = 0\ \text{ for } y = \pm 1/2. \end{gathered} \tag{5.1}$$

Here $\mathrm{R} = \mathrm{Gr}\,\mathrm{Pr}$ is the Rayleigh number, $\mu = \mathrm{Gv}\,\mathrm{Pr}\,m/b$ is the vibration Rayleigh number, which characterizes convection in the porous medium under weightlessness [10], $D \equiv d/dy$; $L \equiv D^2 - \alpha^2$. We will demonstrate that in the unstable case all the increments σ are real and the instability threshold ($\sigma_r = 0$) is determined by the equation $\sigma = 0$, which expresses the monotonicity principle. We multiply the first expression of Eq. (5.1) by θ^* and the second by w^*, then integrate each of the equations thus obtained over y from $-1/2$ to $1/2$ (the asterisk denotes the complex conjugate). Then

$$\mathrm{Pr}cbI_1\sigma^2 + (\mathrm{Pr}bI_1 + cI_2)\sigma + I_2 - (\mathrm{R} + \mu\cos^2\varphi)\alpha^2 I_3 + \mu I_4 = 0; \tag{5.2}$$

$$I_5 = \int\limits_{-1/2}^{1/2}(\alpha^2\sin\varphi\theta + i\alpha\cos\varphi D\theta)\,w^*\,dy. \tag{5.3}$$

TABLE 1

μ	$\alpha^2_{m,1}$	$R_{m,1}$	$\alpha^2_{m,2}$	$R_{m,2}$	$\alpha^2_{m,1}\mu^{1/2}$	$R_{m,1}\mu^{-1/2}$
0	9,8700	39,48	39,48	157,91		
10	8,7030	44,32	38,25	162,87		
20	7,7100	48,86	37,06	167,76		
50	5,6530	60,83	33,75	181,93		
100	3,9810	76,93	29,11	204,05		
200	2,6840	101,48	22.61	243,33		
800	1,2220	120,85	10,73	405,91		
1 800	0,7870	277,23	6,74	579,43		
4 800	0,4680	445,65	3,92	914,03		
8 000	0,3590	572,21	2.97	1166,40	32,11	6,397
10 000	0,3200	638,51	2,64	1298,80	31,99	6,385
14 500	0,2643	766,73	2,17	1554,80	31,82	6,367
100 000	0,0990	1996,90	0,80	4014,10	31,32	6,315
1 000 000	0,0310	6293,10	0,25	12606,00	31,10	6,293
2 000 000	0,0220	8896,00	0,18	17811,00	31,08	6,290
10 000 000	0,0098	19879,00	0,08	39778,00	31,04	6,286

Here

$$I_1 = \int_{-1/2}^{1/2} (|D\theta|^2 + \alpha^2|\theta|^2)\,dy;$$

$$I_2 = \int_{-1/2}^{1/2} |L\theta|^2\,dy;\quad I_3 = \int_{-1/2}^{1/2} |\theta|^2\,dy;$$

$$I_4 = \int_{-1/2}^{1/2} (\alpha^2 \sin\varphi w + i\alpha\cos\varphi Dw)\,\theta^*\,dy;$$

$$I_5 = \int_{-1/2}^{1/2} (|Dw|^2 + \alpha^2|w|^2)\,dy.$$

From Eq. (5.3) we easily find that $I_4 = I_5$. Hence all the integrals appearing in Eq. (5.2) are positive. Taking $\sigma = \sigma_r + i\sigma_i$ and separating real and imaginary components, we obtain

$$\begin{gathered}\sigma_i(2\Pr cbI_1\sigma_r + \Pr bI_1 + cI_2) = 0,\\ \Pr cbI_1(\sigma_r^2 - \sigma_i^2) + (\Pr bI_1 + cI_2)\sigma_r + I_2 - (\mathrm{R} + \mu\cos^2\varphi)\,\alpha^2 I_3 + \mu I_5 = 0,\end{gathered} \tag{5.4}$$

from which it follows that $\sigma_i \neq 0$ only for $\sigma_r < 0$ and the instability threshold is attained at $\sigma = 0$. Moreover, it follows from Eqs. (5.4) that for transverse vibration ($\varphi = \pi/2$) filtration convection in a liquid layer under weightlessness is impossible.

6. Vertical Oscillations. It follows from Sec. 5 that for vertical oscillations the monotonicity principle is satisfied for all values of b and m, while for the amplitude $\theta(y)$ we have

$$L^3\theta - \mathrm{R}\alpha^2 L\theta - \mu\alpha^4\theta = 0,\ \theta = L\theta = L^2\theta = 0 \text{ for } y = \pm 1/2,$$

which admits the exact solution $\theta(y) = \sin\pi n(y + 1/2)$. We write the equation of the neutral curve in the plane (R, α) in the form

$$\mathrm{R}(\alpha) = \frac{(\pi^2 n^2 + \alpha^2)^2}{\alpha^2} + \mu\frac{\alpha^2}{\pi^2 n^2 + \alpha^2}. \tag{6.1}$$

From this relationship we find that high speed vertical oscillations hinder development of convection in a porous medium layer. We will now clarify the behavior of the minimum with respect to α of the Rayleigh number $R_m(n, \mu)$. The condition for the minimum $\partial R/\partial\alpha = 0$ leads to the equation

$$x^4 + 2\beta x^3 + \mu\beta x^2 - 2\beta^3 x - \beta^4 = 0\ (x = \alpha^2,\ \beta = \pi^2 n^2),$$

which for fixed β and μ has one positive root, with the conditions $x < \beta$, $\partial x/\partial\beta > 0$, $\partial x/\partial\mu < 0$ being satisfied. Consequently, the critical wave number α_m increases with n and decreases with μ. For large values of the vibration Reynolds number ($\mu \to \infty$) we can construct the asymptote

$$R_{m,n} = 2\pi n \mu^{1/2} + \pi^2 n^2 + O(\mu^{-1/2}),$$
$$\alpha^2_{m,n} = \pi^3 n^3 \mu^{-1/2} + o(\mu^{-1}).$$

The calculations shown in Table 1 indicate that beginning with $\mu = 10^5$ asymptotic values are reached.

7. Absolute Stabilization. We will show that in the case of vertical oscillations the state of relative rest may be stable for any temperature gradient. To do this we write $\mu = R^2 r$ [$r = a^2\chi\nu/(2g^2\ell^2 Kbm^{-1})$ is the vibration parameter, which is temperature-independent]. Then from Eq. (6.1) we obtain the equation of the relative Rayleigh number

$$r\alpha^4 R^2 - \alpha^2(\pi^2 n^2 + \alpha^2)R + (\pi^2 n^2 + \alpha^2)^3 = 0,$$

the roots of which give the equations of the neutral curves

$$R(n, \alpha) = \frac{\pi^2 n^2 + \alpha^2}{2r\alpha^2}\left(1 \mp \sqrt{1 - 4r(\pi^2 n^2 + \alpha^2)}\right), \tag{7.1}$$

which have the form of "tongues." The base instability level is reached at n = 1. Moreover, it follows from Eq. (7.1) that for r there exists a limiting value $r_* = 1/(4\pi^2)$, upon attainment of which absolute stabilization sets in. This means that the vibration rate must satisfy the condition

$$a \geqslant gl\pi^{-1}(Kbm^{-1}/(2\chi\nu))^{1/2}.$$

We note that the conclusions made are valid for vibration of sufficiently high frequency, allowing use of the averaging method. The author thanks V. I. Yudovich for formulating the problem and his valuable remarks.

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